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Large spin limits of AdS/CFT and generalized Landau-Lifshitz equations

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Abstract: We consider $AdS_5 \times S^5$ string states with several large angular momenta along AdS_5 and S^5 directions which are dual to single-trace Super-Yang-Mills (SYM) operators built out of chiral combinations of scalars and covariant derivatives. In particular, we focus on the SU(3) sector (with three spins in S^5) and the SL(2) sector (with one spin in AdS_5 and one in S^5), generalizing recent work [1] and [2] on the SU(2) sector with two spins in S^5. We show that, in the large spin limit and at leading order in the effective coupling expansion, the string sigma model equations of motion reduce to matrix Landau-Lifshitz equations. We then demonstrate that the coherent-state expectation value of the one-loop SYM dilatation operator restricted to the corresponding sector of single trace operators is also effectively described by the same equations. This implies a universal leading order equivalence between string energies and SYM anomalous dimensions, as well as a matching of integrable structures. We also discuss the more general 5-spin sector and comment on SO(6) states dual to non-chiral scalar operators.

Keywords: Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

Following earlier suggestions [3, 4] to study sub-sectors of string states with large quantum numbers, it was proposed in [5, 6] (see also [7] for a review) that spinning string states with 2+3 angular momenta $(S_1, S_2; J_1, J_2, J_3)$ in $AdS_5 \times S^5$should be dual to local operators[1]

$$O_{J,S} = \mathrm{Tr}(D_{1+i2}^{S_1} D_{3+i4}^{S_2} \Phi_1^{J_1} \Phi_2^{J_2} \Phi_3^{J_3}) + \cdots \tag{1.1}$$

in the large N limit of the maximally supersymmetric Super-Yang-Mills (SYM) theory on R^4. In particular, the classical string energy

$$E = J + S + c_1 \frac{\lambda}{J} + c_2 \frac{\lambda^2}{J^3} + \cdots\,, \qquad S = S_1 + S_2\,, \qquad J = J_1 + J_2 + J_3\,, \tag{1.2}$$

which happens to have a regular expansion in powers of $\tilde{\lambda} = \frac{\lambda}{J^2}$ at large total S^5 angular momentum J was suggested to reproduce the anomalous dimensions of the corresponding

[1]Here and in similar expressions below $+\ldots$ stands for all other orderings of the fields and derivatives inside the trace.

operators computed in the same limit

$$J \to \infty\,, \qquad \tilde{\lambda} = \frac{\lambda}{J^2} = \text{fixed}\,. \tag{1.3}$$

In the "SU(2)" sub-sector of 2-spin states represented by strings positioned at the center of AdS_5 and rotating in two out of three planes, i.e. having $S_1, S_2 = 0, J_3 = 0,\ J_1, J_2 \neq 0$, this proposal was explicitly confirmed in [8]–[11] by comparing the leading coefficient $c_1 = c_1(\frac{J_1}{J_2})$ in the string energy for particular 2-spin string solutions with the one-loop anomalous dimensions of the scalar SYM operators $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}) + \cdots$. The anomalous dimensions were computed [8, 11] using the SU(2) Heisenberg $\mathrm{XXX}_{\frac{1}{2}}$ spin chain interpretation of the one-loop anomalous dimension in the corresponding scalar sector [12] and taking a thermodynamic $J \to \infty$ limit of the Bethe ansatz solution for the eigenvalues ($\frac{1}{J}$ corrections should correspond to quantum string sigma model corrections). This "one-loop" agreement was extended to the "two-loop" (c_2-coefficient) level [13] and also demonstrated (at one and two loop orders) for all classical solutions [14] using integrable model/Bethe ansatz techniques. A similar approach was applied also to theories with less supersymmetry [15] and to examples with both open and closed strings [16].

A more universal and potentially deeper understanding of how this relation between (classical) string theory and (quantum) gauge theory arises in the (J_1, J_2) sector was recently presented in [1, 2]. In this approach one identifies a collective coordinate α associated to J and eliminates it from the dynamics. As a result, the action of the classical bosonic $AdS_5 \times S^5$sigma model can be rewritten in the limit $J \to \infty,\ \tilde{\lambda} < 1$ as a non-relativistic two-dimensional theory for the "transverse" string coordinates $n_i \quad (n^2 = 1)$, with the structure

$$L = J\left(C_i(n)\partial_0 n_i - \left[\tilde{\lambda} a_0(\partial_1 n)^2 + \tilde{\lambda}^2[a_1(\partial_1^2 n)^2 + a_2(\partial_1^2 n)^4] + \cdots\right]\right), \tag{1.4}$$

where $C_0 \equiv C_i(n)\partial_0 n_i$ may be interpreted as a WZ-type term (C_i is a monopole potential, $dC_i = -\frac{1}{2}\epsilon_{ijk}n_j dn_j$). It was shown that this action agrees precisely (at order $\tilde{\lambda}$ [1] and $\tilde{\lambda}^2$ [2]) with the corresponding low-energy effective action of the SU(2) ferromagnetic spin chain with the hamiltonian H given by the sum of the one-loop [12] and two-loop [17] dilatation operators. The leading term in the latter action is determined [18, 19] by the coherent state [20] ($\langle n|\,\sigma_i\,|n\rangle = n_i$) expectation value of H. The agreement at the level of two-dimensional actions implies a matching between energies of all string/spin chain solutions and gives a direct relation between integrable structures (observed earlier using Bethe ansatz approach in [21, 22, 14]).

It is of obvious interest to extend the approach of [1, 2] to other sectors of rotating string states. To do this one has to identify subsectors of operators of the gauge theory which are closed under renormalization at least at one-loop. Here we will be interested only in the bosonic subsectors. Apart from the SU(2) sector which is closed to all loop orders, other such sectors are[2]

[2]We are grateful to Niklas Beisert for a discussion and clarification of this issue.

(i) the three-spin "SU(3)" sector of string configurations with all three S^5 angular momenta (J_1, J_2, J_3) being non-zero. These are dual to more general chiral operators $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3}) + \cdots$ (which form a set closed only under one-loop renormalization [17])

(ii) the two-spin "SL(2)" sector of string configurations with one AdS_5 spin ($S = S_1$) and one S^5 angular momentum ($J = J_3$), which are dual to operators $\mathrm{Tr}(D_{1+i2}^S\Phi^J) + \cdots$ (forming a set closed under renormalization to all orders [23]–[25]).

(iii) the three-spin "SU(1,2)" sector of string configurations with two AdS_5 angular momenta (S_1, S_2) and one S^5 spin ($J = J_3$), which are dual to operators $\mathrm{Tr}(D_{1+i2}^{S_1}D_{3+i4}^{S_2}\Phi^J) + \cdots$ (which form a set closed under one-loop renormalization).

Operators in other sectors carrying more general configurations of non-zero spins $(S_1, S_2, J_1, J_2, J_3)$ mix with fermionic operators already at one loop and would require to consider superspin chains [24, 26] and to include fermions on the string sigma-model side.

Earlier results demonstrating the matching of the leading c_1 coefficient in the string energy for certain string solutions and the corresponding one-loop anomalous dimensions were already found in the SU(3) case in [22, 27] (see also [28]) and in the SL(2) case in [11].

In the present paper we show that for the SU(3) and SL(2) sectors, the one-loop equivalence holds universally at the level of the corresponding effective two-dimensional actions. This implies a manifest agreement of the leading-order coefficients in the classical string energy and in the one-loop SYM anomalous dimensions for all possible configurations with given charges and also guarantees a matching of other conserved charges (i.e. the equivalence of integrable structures). The equations that follow from the resulting leading-order 2-d action are matrix generalizations of the Landau-Lifshitz equations.

On the string side (section 2), we use a Hopf-type parametrization of AdS_5 and S^5 metrics separating a single common phase direction. On the SYM side, in the SU(3) sector (section 3) our starting point will be the general expression for the one-loop dilatation operator in the scalar sector as a hamiltonian of an SO(6) spin chain [12] which we restrict to the chiral SU(3) states and compute its expectation value in the SU(3) coherent state formalism. In the SL(2) sector (section 4) we use the expression for the one-loop dilatation operator as a hamiltonian of the $\mathrm{XXX}_{-\frac{1}{2}}$ SL(2) spin chain derived in [23]. Here the length of the spin chain is J, but the number of states S at each site can be arbitrarily large. We define the relevant coherent state and use it to find the associated semi-classical action by computing the expectation value of the spin chain hamiltonian.

Both SU(3) and SL(2) dilatation operators are special cases of the most general PSU(2,2|4) one-loop dilatation operator [24, 26]. In the general sector involving non-chiral operators one does not expect a direct semi-classical relation to string theory: string α' corrections are expected to be important in this case even in the $J \to \infty$ limit so one should be comparing to the full quantum string theory. Still, it may be of interest to study a sigma model that represents a semi-classical coherent state effective action of the PSU(2,2|4) spin chain. In section 5 we comment on the coherent state expectation value

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of the SO(6) spin chain hamiltonian representing the one-loop dilatation operator in the sector of general (non-chiral) scalar operators [12]. Some conclusions will be summarized in section 6.

In appendix A we present a lightning review of coherent states following [20] and discuss more explicitly the coherent states for SO(6). Some of the computational details relevant to section 4 are presented in appendix B.

2. From $AdS_5 \times S^5$string sigma model to Landau-Lifshitz equations

In this section we give a procedure for "rearranging" the classical action of $AdS_5 \times S^5$string sigma model in the large S^5 spin limit which generalizes the one in [1, 2] from the 2-spin sector to more general configurations with two AdS_5 spins and three S^5 spins. It leads to a non-relativistic 2-d action for "transverse" string coordinates which is first order in time derivatives and higher order in spatial derivatives. The equations of motion that follow from the leading-order term in this action are the generalized Landau-Lifshitz equations.

2.1 Parametrization of $AdS_5 \times S^5$

As in [5], we parametrize the $AdS_5 \times S^5$ metric in terms of 3+3 complex coordinates (we assume summation over repeated indices $i, j = 1, 2, 3$ and * denotes complex conjugation)

$$ds^2 = dY_i^* dY^i + dX_i^* dX_i \,, \tag{2.1}$$

where $Y^i = \eta^{ij} Y_j$ with $\eta^{ij} = \mathrm{diag}(-1, 1, 1)$ and

$$Y_i^* Y^i = -1 \,, \qquad X_i^* X_i = 1 \,. \tag{2.2}$$

Introducing new coordinates y, α, V_i, U_i as follows

$$Y_i = e^{iy} V_i \,, \qquad X_i = e^{i\alpha} U_i \,, \qquad V_i^* V^i = -1 \,, \qquad U_i^* U_i = 1 \,, \tag{2.3}$$

the metric becomes

$$\begin{aligned} ds^2 &= -(dy + B)^2 + dV_i^* dV^i + B^2 + (d\alpha + C)^2 + dU_i^* dU_i - C^2 \\ &= -(dy + B)^2 + (d\alpha + C)^2 + D^* V_i^* D V^i + D^* U_i^* D U_i \,, \end{aligned} \tag{2.4}$$

where

$$B = i V_i^* dV^i \,, \qquad DV_i = dV_i - iBV_i \,, \qquad C = -iU_i^* dU_i \,, \qquad DU_i = dU_i - iCU_i \,. \tag{2.5}$$

Here y and α are "overall" phases of AdS_5 and S^5 and U_i and V_i are projective space coordinates: the metric is invariant under a simultaneous shift of y and rotation of V_a, as well as a shift of α and rotation of U_i. The U(1) connections B and C are real. This parametrization corresponds to a Hopf U(1) fibration of S^5 over CP^2 and a similar fibration of AdS_5 over a non-compact version of CP^2. Indeed, $ds_4^2 = DU_i^* DU_i$ is the Fubini-Studi metric on CP^2 and $K = \frac{1}{2} dC$ is the covariantly constant Kähler form on CP^2.[3]

[3]For other applications of similar parametrizations see, e.g., [29, 30].

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It is useful also to recall the relation to the standard angular parametrization of $AdS_5 \times S^5$: $(Y_i = (Y_0, Y_1, Y_2),\ X_i = (X_1, X_2, X_3))$

$$Y_0 = \cosh\rho\ e^{it}\,, \qquad Y_1 = \sinh\rho\ \cos\theta\ e^{i\phi_1}\,, \qquad Y_2 = \sinh\rho\ \sin\theta\ e^{i\phi_1}\,, \tag{2.6}$$
$$X_1 = \sin\gamma\ \cos\psi\ e^{i\varphi_1}\,, \qquad X_2 = \sin\gamma\ \sin\psi\ e^{i\varphi_2}\,, \qquad X_3 = \cos\gamma\ e^{i\varphi_3}\,. \tag{2.7}$$

2.2 Large spin limit of $AdS_5 \times S^5$sigma model

Let us now consider the string sigma model action for the metric (2.4):

$$L = -\frac{1}{2}\left[-(\partial_a y + B_a)^2 + (\partial_a \alpha + C_a)^2 + D_a^* V_i^* D^a V^i + D_a^* U_i^* D^a U_i\right]\,. \tag{2.8}$$

It is clear from (2.6),(2.7) that the conserved charge corresponding to translations in y is the difference (or sum, depending on conventions) of AdS energy and two AdS spins, i.e. $E - (S_1 + S_2)$, while the charge corresponding to α is the total S^5 angular momentum $J \equiv J_1 + J_2 + J_3$ (see [5]). Consider string configurations for which $y \approx \alpha$, in other words interpret y as time and boost along α. Such configurations carry large spin with $E = J + S + O(\frac{\lambda}{J})$. Note that we are using the existence of one common "angle" in AdS_5 (y) and one in S^5 (α) which enter the metric(2.4) with the *opposite* signs. This procedure does not apply if all the S^5 directions are trivial, i.e. if the string were moving only in AdS_5. Indeed, in this case it is known [4, 31, 5] that the string energy does not have a regular expansion in λ at large spin; the discussion that follows applies only to string configurations with at least one angular momentum component in S^5 direction.

Below we will be interested only in the leading "one-loop" term in the large spin or small $\tilde\lambda = \frac{\lambda}{J^2}$ expansion of the action. The reason is that, as already mentioned above, while the two-spin operators from the SU(2) sector discussed in [1, 2] (and also two-spin operators from SL(2) sector) are closed under renormalization to all loop orders, this is not so for more general 3-spin operators, i.e. one is able to compare to (one-loop) SYM theory only the leading-order term in expansion of the sigma model action.

Following [1, 2], we gauge away the "longitudinal" coordinates y and α and arrive at an action for the "transverse" coordinates V_i and U_i (which, in particular, determine a "shape" of rigid rotating strings in the solutions discussed in [5, 9, 10, 32]). In the general case of the (S, J) sector of configurations with all 5 spins non-zero the choice of a useful gauge fixing procedure appears to be non-trivial (see equation (2.12) below). One possibility is to start with a first-order form of the action as in [2] and fix the momenta corresponding to y and α to be homogeneous. Since we are interested only in the leading order correction it should be sufficient to use the conformal gauge supplemented by a condition like $y = \kappa\tau$ or $\alpha = \mathcal{J}\tau$ (analogous to $x^+ = p^+\tau$) that fixes the remaining conformal diffeomorphisms. The difficulty in choosing such a simple gauge for the general case of four- or five-spin configurations should be effectively related to the fact (mentioned in the Introduction) that the corresponding more general gauge theory operators built out of chiral scalars and covariant derivatives do not form a closed subsector mix with operators involving fermions already at one loop.

Instead of attempting to address the most general case of all S_n and J_i being non-zero here we shall concentrate on two special three-spin cases (related by an analytic continuation [11]):

(i) $S_1, S_2 = 0,\ J_1, J_2, J_3 \neq 0,$ i.e. $Y_0 = e^{it},\ Y_1, Y_2 = 0,\ y = t$

(ii) $J_1, J_2 = 0,\ S_1, S_2, J_3 \neq 0,$ i.e. $X_1, X_2 = 0,\ X_3 = e^{i\varphi_3},\ \alpha = \varphi_3$

In the first case we consider string configurations with $Y_1, Y_2 = 0$ and $y = t$ (so that $B_a = 0$) while in the second case we assume $X_1, X_2 = 0$ and thus $\alpha = \varphi_3$ (and $C_a = 0$). In the first case we may apply a boost, i.e. change the coordinates so that

$$v \equiv y = t\,, \qquad u \equiv \alpha - y\,. \tag{2.9}$$

The sigma model lagrangian corresponding to the metric (2.4) then becomes

$$L = -\partial^a v(\partial_a u + C_a) - \frac{1}{2}(\partial_a u + C_a)^2 - \frac{1}{2} D^a U_i^* D_a U_i\,, \tag{2.10}$$

where $C_a = -iU_i^* \partial_a U_i$. We may choose the conformal gauge and supplement it with the condition

$$v = \kappa\tau\,, \tag{2.11}$$

since this satisfies the equation of motion $\partial^2 v = 0$.

As an aside, we note that in the more general case of non-zero U_i and V_i the equations of motion for u and v would be

$$\partial^2 u + \partial^a(C_a - B_a) = 0\,, \qquad \partial^2 v + \partial^a B_a = 0\,. \tag{2.12}$$

In that case one could choose $v = \kappa\tau$ only when $\partial^a B_a = 0$.

Returning to the $V_0 = 1,\ V_1 = V_2 = 0$ case, we may solve the conformal gauge constraints

$$\begin{aligned} 0 &= \partial_0 v(\partial_1 u + C_1) + \partial_1 v(\partial_0 u + C_0) + (\partial_0 u + C_0)(\partial_1 u + C_1) + \\ &\quad + \frac{1}{2}(D_0 U_i^* D_1 U_i + \text{c.c.})\,, \end{aligned} \tag{2.13}$$

$$\begin{aligned} 0 &= 2\partial_0 v(\partial_0 u + C_0) + 2\partial_1 v(\partial_1 u + C_1) + (\partial_0 u + C_0)^2 + (\partial_1 u + C_1)^2 + \\ &\quad + D_0 U_i^* D_0 U_i + D_1 U_i^* D_1 U_i\,, \end{aligned} \tag{2.14}$$

for $D_a u \equiv \partial_a u + C_a$ and eliminate u from the dynamics, getting an effective action for U_i only. As follows from the constraints, to leading order (see also [2])

$$\kappa \approx \mathcal{J} = \frac{1}{\sqrt{\tilde\lambda}}\,, \qquad J = \sqrt{\lambda}\mathcal{J}\,, \qquad \tilde\lambda \equiv \frac{\lambda}{J^2}\,. \tag{2.15}$$

Note also that $\partial^a v(\partial_a u + C_a) = -\kappa C_\tau +$total derivative, where C_τ is linear in time derivatives of U_i. To develop a $\kappa \to \infty$ or $\tilde\lambda \to 0$ expansion it is natural to re-scale [2] the τ coordinate by κ, introducing new time coordinate t

$$\tau = \kappa \mathrm{t}\,, \qquad t = \kappa\tau = \kappa^2 \mathrm{t} \approx \tilde\lambda^{-1}\mathrm{t}\,. \tag{2.16}$$

Then to the leading order in $\tilde{\lambda}$ the action is[4]

$$I = \sqrt{\lambda} \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi}\, L = J \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \tilde{L} \,, \qquad \tilde{L} = \tilde{L}^{(0)} + O(\tilde{\lambda}) \,, \tag{2.17}$$

$$\tilde{L}^{(0)} = C_0 - \frac{1}{2}|D_1 U_i|^2 \,, \qquad C_0 \equiv -iU_i^* \partial_t U_i \,. \tag{2.18}$$

In the two-spin case ($U_3, J_3 = 0$) considered in [1, 2] the equations that follow from (2.18) are the standard Landau-Lifshitz (LL) equations for a classical ferromagnet.[5] To see this explicitly, define a unit vector [2]

$$n_i = \mathbf{U}^\dagger \sigma_i \mathbf{U} \,, \qquad \mathbf{U} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \qquad n_i n_i = 1 \,, \tag{2.19}$$

where σ_i are Pauli matrices. Then (2.18) becomes

$$\tilde{L}^{(0)} = \mathcal{L}_{\rm WZ}(n) - \frac{1}{8}(\partial_1 n_i)^2 \,, \tag{2.20}$$

where

$$\mathcal{L}_{\rm WZ} = C_0 = -\frac{1}{2}\int_0^1 dz\ \varepsilon^{ijk} n_i \partial_z n_j \partial_0 n_k \,. \tag{2.21}$$

The corresponding equations of motion are the usual Landau-Lifshitz equations

$$\partial_0 n_i = \frac{1}{2}\varepsilon_{ijk} n_j \partial_1^2 n_k \,. \tag{2.22}$$

In the second case (which is also one-loop closed on the gauge theory side [24, 11]) where α is a decoupled coordinate (like t was in the first case) it is natural to set instead of (2.9)

$$u \equiv \alpha \,, \qquad v \equiv y - \alpha \,, \tag{2.23}$$

so that

$$L = \partial^a u(\partial_a v + B_a) + \frac{1}{2}(\partial_a v + B_a)^2 - \frac{1}{2}D_a^* V_i^* D^a V^i \,. \tag{2.24}$$

Then choosing conformal gauge supplemented by

$$u = \alpha = \mathcal{J}\tau \,, \tag{2.25}$$

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[4]Note that in the $\kappa \to \infty$ limit the $\tau\tau$ component of the induced metric approaches zero, i.e. the world sheet metric (in the conformal gauge) degenerates (as was first noted in a special 2-spin case in [33] and was further applied in [34]).

[5]Let us note that in the case when only one out of the three U_i's is non-zero, i.e. when we have only one component of the angular momentum as in the point-like geodesic case, then $U_1 = 1,\ U_2, U_3 = 0, C_a = 0$ and so the action in (2.10) becomes trivial: $L = -\partial^a t \partial_a u - \frac{1}{2}(\partial_a u)^2$. One may study, however, quadratic fluctuations near such a trajectory. The action that summarizes them can be found by setting $Y_0 = a e^{it}$, $a^2 = 1 + |Y_s|^2$, $Y_s = \epsilon \tilde{Y}_s$, and $X_1 = r e^{i(t+\epsilon^2 u)}$, $r^2 = 1 - |X_s|^2$, $X_s = \epsilon \tilde{X}_s$ $(s = 1, 2)$. Expanding in $\epsilon \to 0$ we get from $ds^2 = -a^2 dt^2 + r^2 d\alpha^2 + dr^2 + da^2 + |dX_s|^2 + |dY_s|^2$: $L = -\frac{1}{2}\epsilon^2 \Big[2\partial^a t \partial_a u - (|\tilde{X}_s|^2 + |\tilde{Y}_s|^2)\partial^a t \partial_a t + |\partial_a X_s|^2 + |\partial_a Y_s|^2 \Big] + O(\epsilon^4)$. Choosing the "light-cone" gauge $t = \kappa\tau$ (and adding the fermionic part) we are led [31] to the standard quadratic fluctuation action of [35].

and dropping a total derivative term we get

$$L=-\mathcal{J}B_\tau+\frac{1}{2}(\partial_a v+B_a)^2-\frac{1}{2}D_a^*V_i^*D^aV^i\,. \tag{2.26}$$

The conformal gauge constraints again determine $\partial_a v + B_a$ in terms of the "transverse" coordinates V_i. Rescaling the time coordinate as in (2.16)

$$\tau=\mathcal{J}\mathrm{t}\,,\qquad \alpha=\mathcal{J}\tau=\mathcal{J}^2\mathrm{t}=\tilde{\lambda}^{-1}\mathrm{t}\,, \tag{2.27}$$

we end up with a systematic expansion of the sigma model action in powers of $\tilde{\lambda}$ with the leading-order term in the effective lagrangian being

$$\begin{aligned} I &= J\int d\mathrm{t}\int_0^{2\pi}\frac{d\sigma}{2\pi}\,\tilde{L}\,, &\qquad \tilde{L} &= \tilde{L}^{(0)}+O(\tilde{\lambda})\,,\\ \tilde{L}^{(0)} &= -B_0-\frac{1}{2}D_1^*V_iD_1V^i\,, &\qquad B_0 &\equiv iV_i^*\partial_{\mathrm{t}}V^i\,. \end{aligned} \tag{2.28}$$

As C_0 in (2.18), here B_0, which is linear in time derivatives of V_i, plays the role of a WZ-type term in the action that leads to generalized Landau-Lifshitz equations.

It is instructive also to present the explicit form of (2.28) in terms of angular coordinates in the simplest non-trivial case of $S_1\neq 0,\ S_2=0$. Then the relevant part of the $AdS_5\times S^5$metric is (cf. (2.6), (2.7))

$$ds^2=-\cosh^2\rho dt^2+d\rho^2+\sinh^2\rho d\phi_1^2+d\varphi_3^2\,. \tag{2.29}$$

Setting $t=y+\eta,\ \phi_1=-y+\eta,\ \alpha=\varphi_3$ we get

$$ds^2=-(dy+B)^2+d\alpha^2+d\rho^2+\sinh^2 2\rho\ d\eta^2\,,\qquad B=\cosh 2\rho\ d\eta\,. \tag{2.30}$$

The resulting leading term in the effective lagrangian (2.28) is then

$$\tilde{L}^{(0)}=-\cosh 2\rho\ \partial_0\eta-\frac{1}{2}\left[(\partial_1\rho)^2+\sinh^2 2\rho\ (\partial_1\eta)^2\right]. \tag{2.31}$$

This is obviously an analytic continuation of the lagrangian in the (J_1,J_2) (i.e. SU(2)) sector in [1] ($\rho\to i\psi,\ \eta\to-\beta$, see also [11]). Introducing an $\mathrm{SO}(1,2)$ vector

$$\ell_0=\cosh 2\rho\,,\qquad \ell_1=\sinh 2\rho\ \cos 2\eta\,,\qquad \ell_2=\sinh 2\rho\ \sin 2\eta\,, \tag{2.32}$$

which defines a hyperboloid

$$\eta^{ij}\ell_i\ell_j=-1\,,\qquad \eta^{ij}=\mathrm{diag}(-1,1,1)\,, \tag{2.33}$$

we can re-write the lagrangian (2.31) in the same way as (2.20)

$$\tilde{L}^{(0)}=\mathcal{L}_{\mathrm{WZ}}(l)-\frac{1}{8}\eta^{ij}\partial_1\ell_i\partial_1\ell_j\,,\mathcal{L}_{\mathrm{WZ}}=-\frac{1}{2}\int_0^1 dz\ \varepsilon^{ijk}\ell_i\partial_z\ell_j\partial_0\ell_k\,. \tag{2.34}$$

This action is thus a direct "analytic continuation" of the SU(2) action (2.20). The corresponding equation of motion is the analog of the Landau-Lifshitz equation (2.22) with $n_i\to\ell_i$, i.e. with $(-,+,+)$ signature.

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2.3 Matrix Landau-Lifshitz equations

Let us return to the first 3-spin case of $S_1, S_2 = 0,\ J_i \neq 0$ and consider another more explicit form of (2.18) and the generalized LL equations that follow from it. Instead of the unit vector n_i we may also use an SU(2) Lie algebra valued matrix M $(\alpha, \beta = 1, 2)$

$$M_{\alpha\beta} \equiv 2U^*_\alpha U_\beta - \delta_{\alpha\beta}\,, \qquad n_i = \frac{1}{2}\sigma_i^{\alpha\beta} M_{\alpha\beta}\,. \tag{2.35}$$

The matrix $M_{\alpha\beta}$ satisfies the relations

$$\mathrm{Tr} M = 0\,, \qquad M^\dagger = M\,, \qquad M^2 = 1\,, \tag{2.36}$$

with $M^\dagger$ being the hermitean conjugate of M.[6] In terms of M the CP^1 lagrangian $\tilde{L}^{(0)}(U_1, U_2, U_3 = 0)$ becomes

$$\tilde{L}^{(0)}(M) = \mathcal{L}_{\mathrm{WZ}}(M) - \frac{1}{16}\mathrm{Tr}(\partial_1 M \partial_1 M)\,, \tag{2.37}$$

where

$$\mathcal{L}_{\mathrm{WZ}}(M) = \frac{i}{8}\int_0^1 dz\ \mathrm{Tr}(M\,[\partial_z M\,,\,\partial_0 M])\,. \tag{2.38}$$

The equation of motion for M which follows from $\tilde{L}^{(0)}(M)$ is the matrix Landau-Lifshitz equation (see [36])[7]

$$\partial_0 M = -\frac{i}{4}\left[M, \partial_1^2 M\right]\,. \tag{2.39}$$

In the general 3-spin case $(U_3 \neq 0)$ we define an SU(3) Lie algebra valued matrix N $(i, j = 1, 2, 3)$

$$N_{ij} \equiv 3U^*_i U_j - \delta_{ij}\,. \tag{2.40}$$

This matrix satisfies the following constraints (cf. (2.36))

$$\mathrm{Tr}\ N = 0\,, \qquad N^\dagger = N\,, \qquad N^2 = N + 2\,, \tag{2.41}$$

with the last constraint equivalent to $N^{-1} = \frac{1}{2}(N-1)$.[8] The CP^2 lagrangian in equation (2.18) takes the form

$$\tilde{L}^{(0)}(N) = \mathcal{L}_{\mathrm{WZ}}(N) - \frac{1}{36}\mathrm{Tr}(\partial_1 N \partial_1 N)\,. \tag{2.42}$$

and the WZ term is

$$\mathcal{L}_{\mathrm{WZ}}(N) = \frac{i}{18}\int_0^1 dz\ \mathrm{Tr}(N\,[\partial_z N\,,\,\partial_0 N])\,. \tag{2.43}$$

The equation of motion for N is the SU(3) matrix Landau-Lifshitz equation

$$\partial_0 N = -\frac{i}{6}\left[N\,,\,\partial_1^2 N\right]\,. \tag{2.44}$$

[6]These constraints reduce the number of independent real degrees of freedom that M carries from eight to two - the same number as U_1 and U_2 have.

[7]We are grateful to Gleb Arutyunov for bringing to our attention this form of the Landau-Lifshitz equation.

[8]One can shift N by a constant to make it satisfy $N^2 = 1$ at the expense of the zero-trace condition.

Similar expressions are found in the (S,J) case (2.28),(2.31). In this sector we define an SU(1,1) matrix $(\alpha,\beta=0,1)$

$$L_{\alpha\beta} \equiv 2V^*_\alpha V_\beta + \eta_{\alpha\beta}\,, \qquad \eta_{\alpha\beta} = \mathrm{diag}(-1,1)\,. \tag{2.45}$$

Then the equations of motion corresponding to (2.28) can be written as

$$\partial_0 L = -\frac{i}{4}\left[L, \partial_1^2 L\right], \tag{2.46}$$

where matrix products are defined using $\eta^{\alpha\beta}$. Finally, for completness let us note that it is just as easy to write down the generalised Landau-Lifshitz equation for the SU(1,2) sector which corresponds to three non-zero spins (S_1,S_2,J). This equationn is just an analytic continuation of the equation (2.44).

3. From spin chains to sigma models: SU(3) sector

The lagrangians (2.18) and (2.28) following from (2.8) are two "non-relativistic" sigma models with WZ-type terms with the target spaces $CP^2 = \mathrm{U}(3)/[\mathrm{U}(2)\times\mathrm{U}(1)]$ and $\tilde{CP}^2 = \mathrm{U}(1,2)/[\mathrm{U}(2)\times\mathrm{U}(1)]$ respectively. The same expressions (2.18) and (2.28) will appear also on the SYM side as the effective low-energy lagrangians of spin chains associated to the one-loop dilatation operator in the respective sectors. This generalizes the connection [1] between the 1-loop SU(2) spin chain sigma model and the leading-order limit of $AdS_5\times S^5$sigma model from the 2-spin sector (operators $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2})+\ldots$) to more general chiral 3-spin sectors. In the 2-spin SU(2) sector the symmetry group is $G=\mathrm{U}(2)$ and one factorizes [19] over the stability group of a vacuum state, $H=\mathrm{U}(1)\times\mathrm{U}(1)$ getting a sigma model on $G/H=\mathrm{SU}(2)/\,\mathrm{U}(1)=CP^1$. In the more general case of the 3-spin S^5 sector (operators $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3})+\ldots$) we have $G=\mathrm{U}(3)$ and $H=\mathrm{U}(2)\times\mathrm{U}(1)$ so that $G/H=CP^2$.[9]

In this section we find the continuum limit of the coherent state expectation value of the one loop $\mathcal{N}$=4 SYM dilatation operator in the SU(3) sub-sector of operators $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3})+\cdots$. Recall that the one-loop dilatation operator in the scalar SO(6) sector is [12]

$$D_{\mathrm{SO(6)}} = \frac{\lambda}{(4\pi)^2}\sum_{l=1}^{J}\left(K_{l,l+1} + 2 - 2P_{l,l+1}\right), \tag{3.1}$$

where the trace, K, and permutation, P, operators act on $\mathbb{R}^6\otimes\mathbb{R}^6$ as

$$K^{J_l J_{l+1}}_{I_l I_{l+1}} = \delta_{I_l I_{l+1}}\delta^{J_l J_{l+1}}\,, \qquad P^{J_l J_{l+1}}_{I_l I_{l+1}} = \delta^{J_{l+1}}_{I_l}\delta^{J_l}_{I_{l+1}}\,, \tag{3.2}$$

[9] One may wonder if there is a similar limit that would allow one to obtain a more general sigma model that follows from the full SO(6) spin chain (i.e. S^5 sigma model with a WZ term, cf. [19]). It is not clear if that is possible. The SO(6) spin chain contain sectors of non-chiral operators (e.g. $\mathrm{Tr}(\Phi^*\Phi)^n\ldots$), and, a priori, there is no reason to expect to be able to match their 1-loop dimensions with classical string energies, even in the sector of "long" scalar operators. However, there may be still another sub-sector of states for which the matching may be possible — pulsating string states [22]. Their role and place in the present context remains to be understood but it seems that in this case the above "Hopf fibration" parametrization based on separation of one common direction in S^5 and a boost is not a useful one.

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with $\mathbb{R}^6$ being the space of SYM scalars ϕ_I $(I = 1,\ldots,6)$. The restrictions of $D_{\mathrm{SO}(6)}$ to SU(2) and SU(3) sectors can be easily deduced since these sub-sectors are traceless $(\Phi_k = \phi_k + i\phi_{k+3},\ k = 1,2,3)$. In both sectors we have

$$D_{\mathrm{SU}(3)} = \frac{\lambda}{(4\pi)^2}\sum_{l=1}^{J}(2 - 2P_{l,l+1})\,. \tag{3.3}$$

In the SU(2) subsector of SU(3) sector the permutation operator can be expressed in terms of SU(2) generators ($S_i = \frac{1}{2}\sigma_i$ where σ_i are Pauli matrices)

$$P_{l,l+1} = \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{3}\sigma_l^i\sigma_{l+1}^i\,, \tag{3.4}$$

and so

$$D_{\mathrm{SU}(2)} = \frac{\lambda}{(4\pi)^2}\sum_{l=1}^{J}\left(1 - \sum_{i=1}^{3}\sigma_l^i\sigma_{l+1}^i\right). \tag{3.5}$$

Similarly, in the SU(3) sector P can be expressed in terms of the SU(3) algebra generators λ^r, $r = 1,\ldots,8$ (3×3 Gell-Mann matrices)

$$P_{l,l+1} = \frac{1}{3} + \frac{1}{2}\sum_{r=1}^{8}\lambda_l^r\lambda_{l+1}^r\,. \tag{3.6}$$

As a result,

$$D_{\mathrm{SU}(3)} = \frac{\lambda}{(4\pi)^2}\sum_{l=1}^{J}\left(\frac{4}{3} - \sum_{r=1}^{8}\lambda_l^r\lambda_{l+1}^r\right). \tag{3.7}$$

In the following subsections we consider the coherent state expectation values of these operators and determine the associated low-energy effective action.

3.1 The SU(2) subsector

It is useful first to recall the derivation of the continuum limit of coherent state expectation value of $D_{\mathrm{SU}(2)}$ (see [18, 19, 1] and references therein). With S_i denoting the SU(2) generators, the coherent spin state can be defined by applying a rotation $R(n)$ to the highest-weight state oriented along the third axis, which orients it along the unit vector n_i. Equivalently [20], we may define it as

$$|n\rangle \equiv e^{i(n'\times n_0)\cdot S}\,|s,s\rangle = e^{i(n'_y S_x - n'_x S_y)}\,|s,s\rangle\ , \tag{3.8}$$

where $n_0 = (0,0,1)$, $(n')^2 = 1$ and $|s,s\rangle$ is a highest-weight state

$$S_z\,|s,s\rangle = s\,|s,s\rangle\ , \qquad S^2\,|s,s\rangle = s(s+1)\,|s,s\rangle\ . \tag{3.9}$$

The key property of the coherent state $|n\rangle$ is that it satisfies

$$\langle n|\,S_i\,|n\rangle = sn_i\,, \qquad n_in_i = 1\,, \tag{3.10}$$

where n_i parametrizes the coset $\mathrm{SU}(2)/\mathrm{U}(1)$

$$n_1 = \frac{n'_y \sin\Delta}{\Delta}\,, \qquad n_2 = \frac{-n'_x \sin\Delta}{\Delta}\,, \qquad n_3 = \cos\Delta\,, \qquad \Delta = \sqrt{{n'_x}^2 + {n'_y}^2}\,.$$

In order to generalise to other groups it will be useful to work with the matrix M in (2.35) rather than the vector n. The coherent state is then (here we consider the relevant case of $s = \frac{1}{2}$)

$$|M\rangle \equiv e^{i(a\sigma_1 + b\sigma_2)}\,|0\rangle\ , \qquad |0\rangle \equiv \left|\frac{1}{2}, \frac{1}{2}\right\rangle\ , \tag{3.11}$$

where a, b are angular variables. Then

$$\langle M|\,\sigma_i\,|M\rangle = \langle 0|\sum_{j=1}^{3} a_{ij}\sigma_j\,|0\rangle = \langle 0|\,a_{i3}\sigma_3\,|0\rangle = a_{i3}\,, \tag{3.12}$$

where

$$a_{13} = -b\frac{\sin(2\Delta)}{\Delta}\,, \qquad a_{23} = a\frac{\sin(2\Delta)}{\Delta}\,, \qquad a_{33} = \cos(2\Delta)\,, \qquad \Delta = \sqrt{a^2 + b^2}\,. \tag{3.13}$$

The matrix M is

$$M = \sum_{i=1}^{3} a_{i3}\sigma_i\,, \qquad \text{i.e.} \qquad \langle M|\,\sigma_i\,|M\rangle = a_{i3} = \frac{1}{2}\mathrm{Tr}(M\sigma_i)\,. \tag{3.14}$$

Explicitly,

$$M_{\alpha\beta} = 2U^*_\alpha U_\beta - \delta_{\alpha\beta}\ , \tag{3.15}$$

where

$$U_1 = \cos\Delta\ e^{-i\theta/2}\,, \qquad U_2 = \sin\Delta\ e^{i\theta/2}\,, \qquad b + ia \equiv -\Delta e^{i\theta}\,, \qquad \sum_{\alpha=1,2} |U_\alpha|^2 = 1 \tag{3.16}$$

i.e. U_α, and hence M are coordinates on $\mathrm{SU}(2)/\mathrm{U}(1)$.

Next, we may define a coherent state for the whole spin chain as

$$|M\rangle \equiv \prod_{l=1}^{J} |M_l\rangle\ , \tag{3.17}$$

where $|M_l\rangle$ is the coherent state (3.11) at site l. Then

$$\begin{aligned}\langle M|\,D_{\mathrm{SU}(2)}\,|M\rangle &= \frac{\lambda}{(4\pi)^2}\sum_{l=1}^{J}\left[1 - \frac{1}{4}\mathrm{Tr}(M_l\sigma^i)\mathrm{Tr}(M_{l+1}\sigma^i)\right] \\ &= \frac{\lambda}{128\pi^2}\sum_{l=1}^{J}\left[\mathrm{Tr}(M_l\sigma^i) - \mathrm{Tr}(M_{l+1}\sigma^i)\right]^2\ ,\end{aligned} \tag{3.18}$$

i.e.

$$\begin{aligned}\langle M|\,D_{\mathrm{SU}(2)}\,|M\rangle &\to J\int_0^{2\pi}\frac{d\sigma}{2\pi}\,\frac{\tilde{\lambda}}{32}\left[\mathrm{Tr}(\partial_1 M\sigma^i) + O(\frac{1}{J}\partial_1^2 M)\right]^2 \\ &\to J\int_0^{2\pi}\frac{d\sigma}{2\pi}\,\frac{\tilde{\lambda}}{16}\mathrm{Tr}(\partial_1 M\partial_1 M)\,.\end{aligned} \tag{3.19}$$

We have taken a continuum limit defining as in [1, 2] ($0 < \sigma \leq 2\pi$)

$$M(\sigma_l) = M\left(\frac{2\pi l}{J}\right) \equiv M_l\,, \qquad M_{l+1} - M_l = \frac{2\pi}{J}\partial_1 M + \mathcal{O}\left(\frac{1}{J^2}\right), \qquad \partial_1 \equiv \frac{\partial}{\partial\sigma}\,. \tag{3.20}$$

We have assumed that $J \to \infty$ for $\tilde{\lambda} = \frac{\lambda}{J^2}$=fixed. This ensures that terms subleading in $1/J$ drop out. We have also used the completness identity $(\mathrm{Tr}(A\sigma^i))^2 = 2\mathrm{Tr}A^2$ for any traceless 2×2 matrix A, and that $\mathrm{Tr}M_l^2 = 2$, since M can be written in the form (3.15). The full action in the coherent state path integral contains also the WZ term that ensures the correct quantization conditions at each site. After the rescaling of the time $t \to \mathrm{t} = \tilde{\lambda}^{-1}t$ as in [2] the full action becomes the same as in (2.37)

$$I = J\int \mathrm{dt}\int_0^{2\pi}\frac{d\sigma}{2\pi}\,\mathcal{L}_{\mathrm{SU(2)}}\,, \qquad \mathcal{L}_{\mathrm{SU(2)}} = \mathcal{L}_{\mathrm{WZ}}(M) - \frac{1}{16}\mathrm{Tr}(\partial_1 M\partial_1 M)\,. \tag{3.21}$$

This implies, in particular, that the leading order $\tilde{\lambda}$ term in the energy of the 2-spin string solutions agrees [1, 2] with the one-loop term on the SYM side computed in the same $J \to \infty$, $\tilde{\lambda}$=fixed limit (which is thus a semiclassical limit on the spin chain side).

3.2 The SU(3) sector

With the SU(3) generators in the fundamental representation chosen as Gell-Mann matrices λ^r (so that the SU(2) subgroup is generated by $\lambda^1, \lambda^2, \lambda^3$ and the Cartan generators are λ^3 and λ^8) we define the coherent state as

$$|N\rangle \equiv e^{i(a\lambda^4 + b\lambda^5 + c\lambda^6 + d\lambda^7)}\,|0\rangle\ , \tag{3.22}$$

where a, b, c, d are angular variables. The state $|0\rangle$ will be chosen to satisfy (see [19] and also appendix A)

$$\lambda^3\,|0\rangle = h_1\,|0\rangle\ , \qquad \lambda^8\,|0\rangle = h_2\,|0\rangle\ , \tag{3.23}$$

$$\lambda^1\,|0\rangle = 0\,, \qquad \lambda^2\,|0\rangle = 0\,, \tag{3.24}$$

$$\langle 0|\,\lambda^r\,|0\rangle = 0\,, \qquad r = 4, \ldots, 7\,. \tag{3.25}$$

Equation (3.24) implies that the constants h_1, h_2 are

$$h_1 = 0\,, \qquad h_2 = -\frac{2}{\sqrt{3}}\,. \tag{3.26}$$

Then the coherent state $|N\rangle$ satisfies

$$\langle N|\,\lambda_r\,|N\rangle = \langle 0|\sum_{m=1}^{8} b_{rm}\lambda^m\,|0\rangle = \langle 0|\,(b_{r3}\lambda^3 + b_{r8})\lambda^8\,|0\rangle = -\frac{2}{\sqrt{3}}b_{r8} \equiv \frac{2}{3}a_r\,. \tag{3.27}$$

Explicitly, one finds for a_i

$$\begin{aligned}
\frac{2}{3}a_1 &= 2(ac+bd)\frac{\sin^2\Delta}{\Delta^2}\,, & \frac{2}{3}a_2 &= 2(bc-ad)\frac{\sin^2\Delta}{\Delta^2}\,,\\
\frac{2}{3}a_3 &= (a^2+b^2-c^2-d^2)\frac{\sin^2\Delta}{\Delta^2}\,, & \frac{2}{3}a_8 &= -\frac{1}{2\sqrt{3}}(1+3\cos 2\Delta)\,,\\
\frac{2}{3}a_4 &= \frac{b\sin 2\Delta}{\Delta}\,,\quad \frac{2}{3}a_5 = \frac{-a\sin 2\Delta}{\Delta}\,, & \frac{2}{3}a_6 &= \frac{d\sin 2\Delta}{\Delta}\,,\quad \frac{2}{3}a_7 = \frac{-c\sin 2\Delta}{\Delta}\,,
\end{aligned} \tag{3.28}$$

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where now $\Delta = \sqrt{a^2+b^2+c^2+d^2}$. The matrix N, labelling our coherent state $|N\rangle$ is

$$N = \sum_{r=1}^{8} a_r \lambda^r\,, \qquad \text{i.e.} \qquad \langle N|\,\lambda_r\,|N\rangle = \frac{2}{3}a_r = \frac{1}{3}\mathrm{Tr}(N\lambda_r)\,, \tag{3.29}$$

where only four out of the eight real components a_r are independent. This construction guarantees that the matrix N has the following decomposition ($i,j=1,2,3$)

$$N_{ij} \equiv 3U_i^* U_j - \delta_{ij}\,, \qquad \sum_{i=1}^{3} |U_i|^2 = 1\,, \tag{3.30}$$

where U_i are defined without an overall phase. That means that U_i, and hence N, are coordinates on $\mathrm{SU}(3)/(\mathrm{SU}(2)\times \mathrm{U}(1))$ or CP^2. To see this explicitly let us note that any matrix N from SU(3) algebra (3.29) admits a representation (3.30) in terms of U_i if and only if

$$a_1^2 + a_2^2 = \left(1 + a_3 + \frac{a_8}{\sqrt{3}}\right)\left(1 - a_3 + \frac{a_8}{\sqrt{3}}\right), \tag{3.31}$$

$$a_4^2 + a_5^2 = \left(1 + a_3 + \frac{a_8}{\sqrt{3}}\right)\left(1 - 2\frac{a_8}{\sqrt{3}}\right), \tag{3.32}$$

$$a_6^2 + a_2^7 = \left(1 - a_3 + \frac{a_8}{\sqrt{3}}\right)\left(1 - 2\frac{a_8}{\sqrt{3}}\right), \qquad \sum_{r=1}^{8} a_r^2 = 3\,. \tag{3.33}$$

It is straightforward to check that the constants a_r in equations (3.28) satisfy these four equations.

We may then consider the coherent state for the whole spin chain

$$|N\rangle \equiv \prod_{l=1}^{J} |N_l\rangle\ , \tag{3.34}$$

where $|N_l\rangle$ are defined in equation (3.22). Computing the matrix element

$$\begin{aligned}\langle N|\,D_{\mathrm{SU(3)}}\,|N\rangle\ &= \frac{\lambda}{(4\pi)^2}\sum_{l=1}^{J}\left[\frac{4}{3} - \frac{1}{9}\mathrm{Tr}(N_l\lambda^r)\mathrm{Tr}(N_{l+1}\lambda^r)\right] \\ &= \frac{\lambda}{288\pi^2}\sum_{l=1}^{J}\Big[\mathrm{Tr}(N_l\lambda^r) - \mathrm{Tr}(N_{l+1}\lambda^r)\Big]^2\,, \end{aligned}\tag{3.35}$$

and taking the continuum limit with $J\to\infty$, $\tilde{\lambda}$=fixed as in the SU(2) sector we get,

$$\begin{aligned}\langle N|\,D_{\mathrm{SU(3)}}\,|N\rangle\ &\to J\int_0^{2\pi}\frac{d\sigma}{2\pi}\,\frac{\tilde{\lambda}}{72}\left[\mathrm{Tr}(\partial_1 N\lambda^r) + O(\frac{1}{J}\partial_1^2 N)\right]^2 \\ &\to J\int_0^{2\pi}\frac{d\sigma}{2\pi}\frac{\tilde{\lambda}}{36}\mathrm{Tr}(\partial_1 N\partial_1 N)\,. \end{aligned}\tag{3.36}$$

Here we have used that for any traceless 3×3 matrix A

$$(\mathrm{Tr}(A\lambda^r))^2 = 2\mathrm{Tr}A^2\,, \tag{3.37}$$

and that $\mathrm{Tr}N_l^2 = 6$, since N can be written in the form (3.30). Rescaling $t \to \mathrm{t} = \tilde{\lambda}^{-1}t$, the total coherent state path integral action becomes

$$I = J \int \mathrm{dt} \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \mathcal{L}_{\mathrm{SU(3)}}\,, \qquad \mathcal{L}_{\mathrm{SU(3)}} = \mathcal{L}_{\mathrm{WZ}}(N) - \frac{1}{36}\mathrm{Tr}(\partial_1 N \partial_1 N)\,. \tag{3.38}$$

The matrix N satisfies the same constraints as in (2.41). Again, the limit $J \to \infty$, $\tilde{\lambda} =$ fixed is a semiclassical limit on the spin chain side, and the classical action is thus identical to the CP^2 sigma model lagrangian $\tilde{L}^{(0)}(N)$ in equation (2.42).

As a result, we have demonstrated the leading-order equivalence (proposed in [5, 6, 10] and checked previously on particular examples in [22, 27]) between the SYM theory and the string theory in the 3-spin SU(3) sector. This implies in particular the agreement between string energies and anomalous dimensions as well as a relation between integrable structures [21, 22, 28].

4. From spin chains to sigma models: SL(2) sector

Let us now consider the $\mathrm{SL}(2,R)$ sector of the gauge theory [23, 24] containing the operators

$$\mathrm{Tr}(D^S_{1+i2}\Phi^J) + \cdots\,, \tag{4.1}$$

where $\Phi \equiv \Phi_3 = \phi_5 + i\phi_6$ and $D_{1+i2} = D_1 + iD_2$. This subsector is closed under renormalisation in perturbation theory, and is invariant under an SL(2) subalgebra of the superconformal algebra [23]. The planar one-loop anomalous dilatation operator is then found to be equivalent to the hamiltonian of the $\mathrm{XXX}_{-1/2}$ spin chain [23]. The spin chain has J sites, with $D^{n_l}_{1+i2}\Phi$ at each site, $\sum_{l=1}^J n_l = S$, i.e. the "spin variable" at each site transforms in an infinite dimensional $s = -\frac{1}{2}$ representation of SL(2).

This representation can be constructed by standard oscillator methods. Introducing a pair of creation and annihilation operators $a, a^\dagger$ with $[a, a^\dagger] = 1$, one defines the SL(2) generators as

$$\mathrm{S}_0 = a^\dagger a + \frac{1}{2}\,, \qquad \mathrm{S}_- = a\,, \qquad \mathrm{S}_+ = a^\dagger + a^\dagger a^\dagger a\,, \qquad \mathrm{S}_\pm \equiv \mathrm{S}_1 \pm i\mathrm{S}_2\,. \tag{4.2}$$

Then $[a, a^\dagger] = 1$ implies the SL(2) commutation relations[10]

$$[\mathrm{S}_+, \mathrm{S}_-] = -2\mathrm{S}_0\,, \qquad [\mathrm{S}_0, \mathrm{S}_\pm] = \pm\mathrm{S}_\pm\,. \tag{4.3}$$

The SL(2) quadratic Casimir

$$\mathrm{S}^2 = \mathrm{S}_0^2 - \mathrm{S}_1^2 - \mathrm{S}_2^2 = \mathrm{S}_0^2 - \frac{1}{2}(\mathrm{S}_+\mathrm{S}_- + \mathrm{S}_-\mathrm{S}_+) \tag{4.4}$$

in this representaion is equal to $-\frac{1}{4}$, i.e. to $s(s+1)$ for $s = -\frac{1}{2}$. Defining the "highest weight" state $|0\rangle$ as

$$a\,|0\rangle = 0\,, \qquad \mathrm{S}_-\,|0\rangle = 0\,, \qquad \mathrm{S}_0\,|0\rangle = \frac{1}{2}\,, \tag{4.5}$$

[10]More generally, one may consider $\mathrm{S}_0 = a^\dagger a - s$, $\mathrm{S}_- = a$, $\mathrm{S}_+ = -2sa^\dagger + a^\dagger a^\dagger a$, with the SU(2) case corresponding to $\mathrm{S}_+ \to -\mathrm{S}_+$.

we can then construct the representation by associating

$$\frac{1}{n!}(D_{1+i2})^n\Phi \to (a^\dagger)^n\,|0\rangle\ . \tag{4.6}$$

In general [23, 24], the one-loop dilatation operator is

$$D = \frac{2\lambda}{(4\pi)^2}H\,, \qquad H = \sum_{l=1}^{J} H_{l,l+1}\,, \tag{4.7}$$

where H is a spin chain hamiltonian containing nearest neighbour interactions

$$H_{l,l+1} = 2h(\mathrm{S}_{l,l+1}) = \sum_j 2h(j)P_j\,. \tag{4.8}$$

Above, $\mathrm{S}_{l,l+1}$ is the operator that measures total spin at the two sites (i.e. $(\mathrm{S}_l+\mathrm{S}_{l+1})^2$ as the Casimir), P_j projects onto the $\mathrm{S}_{k,k+1}=j$ sector, and $h(j)$ is the j-th harmonic number

$$h(j) \equiv \sum_{k=1}^{j}\frac{1}{k} = \Psi(j+1)-\Psi(1)\,, \tag{4.9}$$

with $\Psi(x)=\Gamma'(x)/\Gamma(x)$. Explicitly, in the present case $H_{l,l+1}$ can be defined by its action on a generic two-site state [23] as

$$H_{l,l+1}(a_l^\dagger)^k(a_{l+1}^\dagger)^{n-k}\,|0,0\rangle = \sum_{p=0}^{n} c_p(k,n)\ (a_l^\dagger)^p(a_{l+1}^\dagger)^{n-p}\,|0,0\rangle\ , \tag{4.10}$$

where

$$[c_p(k,n)]_{k=p} = h(k)+h(n-k)\,, \qquad [c_p(k,n)]_{k\neq p} = -\frac{1}{|k-p|}\,. \tag{4.11}$$

As was found in [23], this H can be interpreted as a hamiltonian of the integrable $\mathrm{XXX}_{-1/2}$ spin chain [37].[11] The definition (4.10) will be sufficient for our present aim of computing the expectation value of $D_{\mathrm{SL}(2)}$ in the corresponding SL(2) coherent state.

The SL(2) coherent state is defined by applying a "rotation" to the highest weight sector that orients the third axis along a unit vector ℓ_i in (2.33). Equivalently, we may define it as [20]

$$|\ell\rangle = e^{i\tau(\sin\phi J_1-\cos\phi J_2)}\,|0\rangle\ , \qquad \text{or} \qquad |\ell\rangle = e^{\zeta J_+}e^{\eta J_0}e^{-\bar\zeta J_-}\,|0\rangle\ , \tag{4.12}$$

where τ and ϕ are two real "angles" related to one complex parameter ζ by

$$\zeta = \tanh\frac{|\tau|}{2}\,, \qquad \eta\equiv\ln(1-|\zeta|^2) = -2\ln\cosh\frac{|\tau|}{2}\,. \tag{4.13}$$

[11]For general s, one can define the hamiltonian of the *integrable* XXX_s spin chain as [37] $\sum_{l=1}^{J}[\Psi(\mathrm{S}_{l,l+1}+1)-\Psi(1)]$ where $\mathrm{S}_{l,l+1}$ is defined in terms of SL(2) spin variables S_l^i (expressed in terms of the oscillators $a_l, a_l^\dagger$ as explained in the previous footnote) through $\mathrm{S}_{l,l+1}(\mathrm{S}_{l,l+1}+1) = 2s(s+1)-2\eta_{ij}\mathrm{S}_l^i\mathrm{S}_{l+1}^j$. Here $\eta_{ij}=(-++)$ (in SU(2) case $-\eta_{ij}=(+++)$).

The second representation of the coherent state is more useful since $J_- |0\rangle = 0$, $J_0 |0\rangle = \frac{1}{2}$ and $J_+^k |0\rangle = \ k!\ a^{\dagger k} |0\rangle$ so that

$$|\ell\rangle = (1 - |\zeta^2|)^{1/2} \sum_{k=0}^{\infty} \zeta^k a^{\dagger k} |0\rangle \ . \tag{4.14}$$

The conjugate coherent state has to satisfy $\langle\ell| |\ell\rangle = 1$ and so is given by

$$\langle\ell| = \langle 0| e^{\bar\zeta J_-} e^{-\eta J_0} e^{-\zeta J_+} = \langle 0| e^{-\zeta J_+} e^{\eta J_0} e^{\bar\zeta J_-} \ . \tag{4.15}$$

Since $\langle 0| J_+ = 0$, $\langle 0| J_- = \frac{1}{2} \langle 0|$, $\langle 0| J_-^k = \langle 0| a^k$, we may write it as

$$\langle\ell| = (1 - |\zeta|^2)^{1/2} \sum_{k=0}^{\infty} \frac{\bar\zeta^k}{k!} \langle 0| a^k \ . \tag{4.16}$$

It is then straightforward to check the basic property of the coherent state

$$\langle\ell| \mathrm{S}_i |\ell\rangle = -\frac{1}{2}\ell_i \,, \qquad \eta^{ij}\ell_i\ell_j = -1 \,, \tag{4.17}$$

where the vector ℓ_i, which parametrizes the hyperboloid $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$, is expressed in terms of ζ by (cf. (2.32), (2.33))

$$\ell_0 = \frac{1 + |\zeta|^2}{1 - |\zeta|^2} \,, \qquad \ell_1 = -\frac{2\zeta}{1 - |\zeta|^2} \,, \qquad \ell_2 = -\frac{2\bar\zeta}{1 - |\zeta|^2} \ . \tag{4.18}$$

Next, we may define the coherent state for the whole spin chain as the product of coherent states at each site, $|\ell\rangle \equiv \prod_{l=1}^{J} |\ell_l\rangle$ and compute

$$\langle\ell| D_{\mathrm{SL}(2)} |\ell\rangle = \frac{2\lambda}{(4\pi)^2} \sum_{l=1}^{J} \langle\ell_{l,l+1}| H_{l,l+1} |\ell_{l,l+1}\rangle \ . \tag{4.19}$$

The result of this computation (with details given in appendix) is remarkably simple:

$$\langle\ell| D_{\mathrm{SL}(2)} |\ell\rangle = \frac{2\lambda}{(4\pi)^2} \sum_{l=1}^{J} \ln \frac{1 - \eta_{ij}\ell_l^i\ell_{l+1}^j}{2} \,, \tag{4.20}$$

or, equivalently,

$$\langle\ell| D_{\mathrm{SL}(2)} |\ell\rangle = \frac{2\lambda}{(4\pi)^2} \sum_{l=1}^{J} \ln \left[1 + \frac{1}{4}\eta_{ij}(\ell_l^i - \ell_{l+1}^i)(\ell_l^j - \ell_{l+1}^j)\right] . \tag{4.21}$$

It is interesting to note that (4.20) is the direct $(-++)$ signature analog on the *classically* integrable lattice hamiltonian for the Heisenberg magnetic [36], which is explicitly given by $\sum_{l=1}^{J} \ln\left(\frac{1+n_l^i n_{l+1}^i}{2}\right)$, where $n^i n^i = 1$ (with $n_3 = \ell_0$, $n_{1,2} = i\ell_{1,2}$).[12]

[12] In contrast, in the SU(2) case with $s = 1/2$ only the continuum (i.e. Landau-Lifshitz) limit of the coherent state expectation value (3.19), i.e. of $\sum_{l=1}^{J}(1 - n_l^i n_{l+1}^i)$, is an integrable classical system. In this case, the fact that both S^5 spins (J_1, J_2) are large, implies that we need to consider large clusters of spins (that exist due to ferromagnetic attraction) which in turn effectively translates into a semiclassical limit.

Note also that since we are interested in comparing to the semiclassical string case, S in (4.1) as well as J should be large, and that, combined with a ferromagnetic nature of the spin chain, effectively corresponds to a low-energy semiclassical limit of the chain. In general, starting with (see footnote 10) $H \sim \sum_{l=1}^{J} \Psi(\mathrm{S}_{l,l+1}+1)$ where $\mathrm{S}_{l,l+1}(\mathrm{S}_{l,l+1}+1) = 2s(s+1) - 2\eta_{ij}\mathrm{S}^i_l\mathrm{S}^j_{l+1}$ and considering a semiclassical limit in which $\mathrm{S}_{l,l+1}$ is large [25], one finds[13]

$$H \sim \sum_{l=1}^{J} \ln \mathrm{S}_{l,l+1} \sim \sum_{l=1}^{J} \ln[2s(s+1) - 2\eta_{ij}\mathrm{S}^i_l\mathrm{S}^j_{l+1}] \, .$$

Then taking the coherent state expectation value assuming all correlators factorize and using that $\langle \mathrm{S}^i_l \rangle = s\ell^i_l$ (see (4.17)) we indeed arrive at (4.20).

The final step is to consider the limit $J \to \infty$ with fixed $\tilde{\lambda} = \frac{\lambda}{J^2}$ which amounts to taking a low-energy continuum limit of this ferromagnetic chain. As in the SU(2) (3.19),(3.20) and SU(3) (3.36) sectors here we set

$$\ell(\sigma_l) = \ell\left(\frac{2\pi l}{J}\right) \equiv \ell_l \, , \qquad \ell_{l+1} - \ell_l = \frac{2\pi}{J}\partial_1 \ell + \mathcal{O}\left(\frac{1}{J^2}\partial_1^2 \ell\right) , \tag{4.22}$$

where $\partial_1 \equiv \partial_\sigma$ derivatives of ℓ are assumed to be finite in the limit. Since we have only one power of λ in (4.20), in expanding the logarithm we need to keep only the order $\frac{1}{J^2}$ term, i.e. the term quadratic in first derivatives. This leads to

$$\langle \ell | \, D_{\mathrm{SL}(2)} \, | \ell \rangle \to J \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[\frac{\tilde{\lambda}}{8} \eta_{ij} \partial_1 \ell^i \partial_1 \ell^j + \mathcal{O}\left(\frac{1}{J^2}\partial_1^4\right) \right] . \tag{4.23}$$

In this way we reproduce the spatial derivative term in the sigma model action (2.34). This implies the general agreement between the string and SYM theories at leading order in $\tilde{\lambda}$ in the SL(2) sector and thus generalizes the previous results [11] for particular solutions.

5. Comments on the non-holomorphic SO(6) sector

In the case of more general sectors involving non-holomorphic operators it is presently not clear how to make a systematic comparison to (a limit of) the string sigma model. However, it may be of interest to repeat the above discussion of the coherent state expectation values of the one-loop dilatation operator in the general SO(6) scalar sector of the operators $\mathrm{Tr}(\phi_{i_1} \cdots \phi_{i_L})$, where ϕ_i are 6 real scalars [12].

In constructing coherent states for a group G one identifies a "vacuum state" $|0\rangle$ together with a (maximal) subgroup H which leaves the vacuum state invariant (see appendix A for a more detailed discussion of the construction of coherent states). This implies that the non-Cartan elements of H annihilate $|0\rangle$. There are four maximal subgroups of SO(6): SO(5), SO(3) $\times$ SO(3), SO(4) $\times$ SO(2) and SU(3) $\times$ U(1). As explained in appendix A, in the case of the fundamental representation of SO(6) there are only two possible choices of H, which admit a suitable vacuum state:

[13] The $s = 1/2$ case is special since here S^i are proportional to Pauli matrices and thus any function of $\mathrm{S}^i_l\mathrm{S}^j_{l+1}$ reduces simply to a linear function ($(\sigma_l \cdot \sigma_{l+1})^2 = 3 - 2\sigma_l \cdot \sigma_{l+1}$, etc.).

(i) $H = \mathrm{SO}(4) \times \mathrm{SO}(2)$ with $|0\rangle_{\mathrm{SO}(4)\times\mathrm{SO}(2)} = (0,0,0,0,1,i)$,

(ii) $H = \mathrm{SO}(5)$ with $|0\rangle_{\mathrm{SO}(5)} = (0,0,0,0,0,1)$.

The first choice corresponds to selecting the BPS operator $\mathrm{Tr}(\phi_5 + i\phi_6)^L$ as a vacuum state. In this case the coset space $G/H = \mathrm{SO}(6)/[\mathrm{SO}(4) \times \mathrm{SO}(2)]$ is an 8-dimensional hermitean symmetric space — the Grassmann manifold $G_{2,6}$ equivalent also to $\mathrm{SU}(4)/S(\mathrm{U}(2)\times \mathrm{U}(2))$.[14] In the second case the vacuum is represented by a non-chiral operator $\mathrm{Tr}(\phi_6)^L$, and $G/H = \mathrm{SO}(6)/\mathrm{SO}(5)$ is S^5. Below we discuss the two cases in turn.

5.1 The $\mathrm{SO}(6)/[\mathrm{SO}(4) \times \mathrm{SO}(2)]$ case

In this section we consider the coherent state $|m\rangle$ for $\mathrm{SO}(6)/[\mathrm{SO}(4) \times \mathrm{SO}(2)]$. The eight dimensional coset space $\mathrm{SO}(6)/(\mathrm{SO}(4)\times\mathrm{SO}(2))$ is spanned by M_{i5} and M_{i6} with $i = 1, \ldots, 4$ (M_{ij} are SO(6) generators in fundamental representation) and so the coherent state is

$$|m\rangle = \exp\left[\sum_{i=1}^{4}(a_i M_{i5} + a_{i+4} M_{i6})\right] |0\rangle \ , \tag{5.1}$$

with $|0\rangle = (0,0,0,0,1,i)$. The state $\langle m|$ is defined by a similar relation so as to satisfy $\langle m|\,|m\rangle = 1$. Introducing an antisymmetric imaginary 6×6 matrix m^{ij}

$$m^{ij} \equiv \langle m|\, M^{ij}\, |m\rangle \ , \tag{5.2}$$

one can check that $\mathrm{Tr}\, m^2 = 2$ and $m^3 = m$, or explicitly

$$\sum_{i,j=1}^{6} m^{ij} m^{ji} = 2\,, \qquad \sum_{k,l=1}^{6} m^{ik} m^{kl} m^{lj} = m^{ij}\,. \tag{5.3}$$

Below we will be interested in $\langle m|\, M^{ij} M^{kl}\, |m\rangle$. On symmetry grounds,

$$\begin{aligned}\langle m|\, M^{ij} M^{kl}\, |m\rangle \ &= \frac{1}{2}(\delta^{il} m^{jk} - \delta^{ik} m^{jl} - \delta^{jl} m^{ik} + \delta^{jk} m^{il}) - \\ &\quad - \frac{1}{2}(\delta^{il} w^{jk} - \delta^{ik} w^{jl} - \delta^{jl} w^{ik} + \delta^{jk} w^{il})\,,\end{aligned} \tag{5.4}$$

where w^{ij} is a symmetric tensor with $\sum_i w^{ii} = 2$ (quadratic Casimir condition). It is possible to show that w^{ij} is the square of m^{ij},

$$w^{ij} = \sum_{k=1}^{6} m^{ik} m^{kj}\,. \tag{5.5}$$

We define the coherent state for the whole spin chain as the product of coherent states at each site $|m\rangle \equiv \prod_{l=1}^{L} |m_l\rangle$, where L is the length of the chain. The one-loop dilatation

[14] In general, the coset $\mathrm{SO}(n)/[\mathrm{SO}(q)\times\mathrm{SO}(n-q)] = G_{q,n}(R)$ is a real Grassmann manifold which consists of all q-dimensional linear subspaces of R^n [39]. In particular, $G_{1,2} = S^1$, $G_{2,4} = S^2 \times S^2$. $G_{2,n}$ spaces are hermitean symmetric spaces.

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operator is proportional to the SO(6) spin chain hamiltonian which is a sum of nearest-neighbour interactions $H_{l,l+1}$. In terms of the SO(6) generators $(M^{ij}_{ab})_l$ at each site, it is given by [12]

$$D_{\rm SO(6)} = \frac{\lambda}{(4\pi)^2}\sum_{l=1}^{L} H_{l,l+1}\,, \qquad H_{l,l+1} = M^{ij}_l M^{ij}_{l+1} - \frac{1}{16}(M^{ij}_l M^{ij}_{l+1})^2 + \frac{9}{4}\,. \tag{5.6}$$

Using equations (5.2) and (5.4) we find that the expectation value of $H_{l,l+1}$ is

$$\begin{aligned}\langle m|\, H_{l,l+1}\,|m\rangle &= 2 + \frac{3}{4} m^{ij}_l m^{ij}_{l+1} - \frac{1}{4} m^{ij}_l m^{jk}_l m^{kl}_{l+1} m^{li}_{l+1}\\ &= \frac{3}{8}{\rm Tr}[(m_l - m_{l+1})^2] + \frac{1}{8}{\rm Tr}[(m_l^2 - m_{l+1}^2)^2]\,,\end{aligned} \tag{5.7}$$

where ${\rm Tr}(m_1 m_2) = m^{ij}_1 m^{ji}_2$, etc., and we used ${\rm Tr} m^2 = 2$ and $m^3 = m$.

To take the continuum limit we again assume that $L \to \infty$ for fixed $\tilde\lambda \equiv \frac{\lambda}{L^2}$. Taylor-expanding as in (3.20), $m_{l+1} = m_l + \frac{2\pi}{L}\partial_1 m + ...$, we need to keep only terms with at most two derivatives (higher derivative terms will be suppressed by $\frac{1}{L}$)

$$\begin{aligned}\langle m|\, D_{\rm SO(6)}\,|m\rangle \to\ & \frac{\lambda}{(4\pi)^2} L\int_0^{2\pi}\frac{d\sigma}{2\pi}\left(\frac{2\pi}{L}\right)^2 \times\\ & \times \left[\frac{3}{8}{\rm Tr}(\partial_1 m)^2 + \frac{1}{4}{\rm Tr}(m^2(\partial_1 m)^2) + \frac{1}{4}{\rm Tr}(m\partial_1 m)^2\right] \qquad (5.8)\\ =\ & L\int_0^{2\pi}\frac{d\sigma}{2\pi}\,\frac{\tilde\lambda}{16}\left[\frac{3}{2}{\rm Tr}(\partial_1 m)^2 + {\rm Tr}(m^2(\partial_1 m)^2) + {\rm Tr}(m\partial_1 m)^2\right], \qquad (5.9)\end{aligned}$$

i.e. (cf. (3.19), (3.36), (4.23))

$$\langle m|\, D_{\rm SO(6)}\,|m\rangle \to L\int_0^{2\pi}\frac{d\sigma}{2\pi}\,\frac{\tilde\lambda}{8}\left[{\rm Tr}(\partial_1 m)^2 + \frac{1}{4}{\rm Tr}(m\partial_1 m)^2\right]. \tag{5.10}$$

Here, we have used the identity ${\rm Tr}(\partial_1 m)^2 = 2{\rm Tr}(m^2(\partial_1 m)^2) + {\rm Tr}(m\partial_1 m)^2$, which follows from $m^3 = m$.

It is possible to rewrite the grassmanian $G_{2,6}$ action corresponding to (5.10) in an equivalent form which is similar to the CP^2 action (2.18) in the SU(3) case (cf. (2.42)). Introducing a complex unit vector V^i $(i = 1,\ldots,6)$ subject also to $V^iV^i = 0$ one can show that a generic imaginary antisymmetric matrix m^{ij} satisfying the constraints (5.3) may be written as[15]

$$m^{ij} = V^iV^{j*} - V^jV^{i*}\,, \qquad V^iV^{i*} = 1\,, \qquad V^iV^i = 0\,. \tag{5.11}$$

The constraints on m^{ij} imply that it has $15 - 1 - 6 = 8$ independent parameters (which is the dimension of $G_{2,6}$). The constraints on V^i leave $12 - 1 - 2 = 9$ real parameters, but in addition m^{ij} is invariant under $V^j \to e^{i\alpha}V^j$, so we may restrict V^i to belong to

[15]The constraint $m^3 = m$ implies that 6×6 matrix m can have eigenvalues equal to $1,-1,1,-1,0,0$ or $1,-1,0,0,0,0$. The latter option is the only possibility in view of ${\rm Tr}m^2 = 2$ condition. Then V^i and V^{i*} are eigenvectors for the eigenvalues 1 and -1.

CP^5, i.e. $G_{2,6}$ is a surface $V^2 = 0$ in CP^5. The components of V^i can be explicitly expressed in terms of 8 real parameters a_n of the coherent state in (5.1). The effective lagrangian corresponding to (5.10) then takes the same form as CP^5 analog of (2.18), i.e. (after rescaling time as in (2.42); note that $\mathrm{Tr}(m\partial_a m)^2 = 0$)

$$\tilde{L}^{(0)} = -iV^{i*}\partial_t V^i - \frac{1}{2}|D_1 V^i|^2 = -iV^{i*}\partial_t V^i - \frac{1}{2}(|\partial_1 V^i|^2 - |V^{i*}\partial_1 V^i|^2)\,, \tag{5.12}$$

with the constraint $V^2 = 0$ (a similar action with an additional constraint $V^{i*}\partial_1 V^i = 0$ was found on the string side in [38]). The SU(3) sector is the special case when $V^a = iV^{a+3} \equiv \frac{1}{\sqrt{2}}U^a$, $a = 1,2,3$, so that $|U^a|^2 = 1$, i.e. U^a belongs to CP^2 and (5.12) reduces to (2.18) or (2.42).

The above coherent state description thus does not capture states with large "extensive" one-loop shift of the dimension $E = L + c_1\lambda L + \cdots$ [12], but instead the most general near-BPS sector of semiclassical string states (including pulsating strings [22]) on S^5 for which one gets again a regular scaling limit, that is a regular dependence of the one-loop correction on $\tilde{\lambda}$.

The precise relation to string theory (in particular, to the sector of pulsating string states) still remains to be understood (for an interesting approach in this direction see [38]).

5.2 The SO(6)/ SO(5) case

In this section we consider the coherent state $|v\rangle$ for SO(6)/ SO(5). The 5-dimensional coset space SO(6)/ SO(5) is spanned by M_{i6} with $i = 1,\dots,5$, and so the coherent states are given by

$$|v\rangle = \exp\left[\sum_{i=1}^{5} a_i M_{i6}\right]|0\rangle\ , \qquad \langle v| = \langle 0|\exp\left[-\sum_{i=1}^{5} a_i M_{i6}\right]. \tag{5.13}$$

As discussed in appendix A, here the vacuum state is $|0\rangle = (0,0,0,0,0,1)$. Since $\langle v|$ is the transpose of $|v\rangle$, the components of the two are indentical. This, together with the fact that the M^{ij} are anti-symmetric matrices, implies that

$$\langle v|\, M^{ij}\, |v\rangle \equiv 0\,, \tag{5.14}$$

for all $i,j = 1,\dots,6$. Next, on symmetry grounds, one has

$$\langle v|\, M^{ij}M^{kl}\, |v\rangle = \delta^{il}v^{jk} - \delta^{ik}v^{jl} - \delta^{jl}v^{ik} + \delta^{jk}v^{il}\,, \tag{5.15}$$

where v^{ij} is a symmetric tensor with $\sum_i v^{ii} = 1$ (quadratic Casimir condition). One can show that

$$v^{ij} = v^i v^j\,, \tag{5.16}$$

with

$$v^i = \frac{a_i \sin\Delta}{\Delta}\,, \qquad i = 1,\dots,5\,, \qquad \text{and} \qquad v^6 = \cos\Delta\,, \qquad \Delta = \sqrt{\Sigma_{i=1}^5\, a_i^2}\,. \tag{5.17}$$

The quadratic Casimir condition now reduces to

$$\sum_{i=1}^{6}(v^i)^2 = 1\,. \tag{5.18}$$

In other words, v^i are coordinates on S^5.

Defining the state of the whole spin chain as the product of coherent states at each site $|v\rangle \equiv \prod_{l=1}^{L}|V_l\rangle$, and using equations (5.14) and (5.15) we find that the expectation value of $H_{l,l+1}$ in (5.6) is

$$\langle v|\,H_{l,l+1}\,|v\rangle = 2-(v^i_l v^i_{l+1})^2 = 1+(v^i_l - v^i_{l+1})^2 - \frac{1}{4}(v^i_l - v^i_{l+1})^4\,. \tag{5.19}$$

If we take the limit $L \to \infty$ we may consider the continuum limit and drop higher derivative terms (assuming that derivatives over σ are fixed in the large L limit). Then (cf. (3.36), (4.23), (5.10))

$$\langle v|\,D_{\rm SO(6)}\,|v\rangle \to \frac{\lambda}{(4\pi)^2}L\int_0^{2\pi}\frac{d\sigma}{2\pi}\left[1+\left(\frac{2\pi}{L}\right)^2(\partial_1 v^i)^2\right]. \tag{5.20}$$

The presence of the first v_i-independent term implies that here we do not get a regular expansion in $\tilde\lambda = \frac{\lambda}{L^2}$. The coherent state expectation value thus captures the large "extensive" one-loop shift of dimensions $E = L + c_1\lambda L + \cdots$ of the type described in [12] which should be characteristic of some oscillator string states. In these cases one expects the full expression for the energy/dimension to contain functions of coupling interpolating from weak to strong coupling: $E_{L\to\infty} = L + f_1(\lambda)L + L^{-1}f_2(\lambda;n_s) + \cdots$, where $f_k(\lambda\to 0) = c_k\lambda + b_k\lambda^2 + \cdots$ and n_s stand for other (oscillator and/or spin) quantum numbers encoded in v_i. One may expect that for states with large quantum numbers $f_k(\lambda\to\infty) = a_k\sqrt{\lambda} + d_k + \cdots$, as, for example, for a single S^5 spin string states [4].

6. Conclusions

In this paper we have demonstrated, in the large spin limit and to leading order in the coupling $\tilde\lambda = \lambda/J^2$, the equivalence of the string theory on $AdS_5\times S^5$ and $N=4$ SU(N) SYM gauge theory in the chiral SU(3) and SL(2) sectors of states. We have developed an expansion of the string sigma model lagrangian, whose leading order term was shown to describe generalizations of the Landau-Lifshitz equations for a classical ferromagnet. On the SYM side, we have computed a coherent state expectation value of the corresponding spin-chain hamiltonian which encodes the one-loop dilatation operator of the theory. In the thermodynamic limit, the resulting coherent-state sigma model matched exactly with the leading order action obtained from the string sigma model lagrangian. In this way we have generalised the recent results of [1, 2] on the SU(2) sector. The matching of the two sigma model lagrangians implies a general agreement (at leading order in $\tilde\lambda$) between the string energies and SYM anomalous dimensions as well as matching of integrable structures, thus generalising previous results in these sectors [31, 10, 11, 22, 28, 21, 27] for particular solutions.

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While the matching of various chiral sectors at leading order in the coupling now seems to be well understood, such an understanding of the more general SO(6) non-chiral sector of operators is still missing. As a step in this direction we computed the continuum limit of the SO(6) spin chain hamiltonian of [12]. In doing this the choice of a vacuum state becomes important. One can choose the vacuum to correspond to the BPS operator $\mathrm{Tr}\Phi^L$, in which case the resulting sigma model has as its target space the 8-dimensional Grassmann manifold SO(6)/[SO(4)×SO(2)]. Then there are no non-derivative leading order corrections to the $E = L$ relation, as should be the case for a sector with a BPS ground state. We expect this sigma model to be related to a gauge theory sector with a regular $\frac{\lambda}{L^2}$ expansion of anomalous dimensions (such as the pulsating string solutions of [22]). Since the target space is 8-dimensional, it is not immediately clear how to relate it to a subsector of the string sigma model. On the other hand, one can choose the vacuum to be represented by a real non-BPS operator $\mathrm{Tr}\phi^L$ whose dimension receives "extensive" (order L) leading order correction. As was shown in section 5.2, this behaviour, typical of more general SO(6) states [12], is indeed captured by the continuum limit of the corresponding coherent- state expectation value of the SO(6) spin chain hamiltonian. Furthermore, in this case the target manifold is indeed S^5 suggesting that a direct relation to the $AdS_5 \times S^5$string sigma model may be possible.

While this paper was nearing completion there appeared an interesting preprint [40] which also extends the SU(2) result of [1] to the SU(3) sector. Our approach is somewhat different (in particular, we use a more covariant parametrization, and exhibit a relation to the matrix Landau-Lifshitz equation) but the final expressions for the actions in that sector agree once expressed in terms of the same coordinates.

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A. General definition of coherent states and SO(6) case

In this appendix we briefly review the definition of coherent states following closely [20]. Given a semisimple group G in the Cartan basis $(\mathrm{H}_i, \mathrm{E}_\alpha, \mathrm{E}_{-\alpha})$ $([\mathrm{H}_i, \mathrm{H}_j] = 0$, $[\mathrm{H}_i, \mathrm{E}_\alpha] = \alpha_i \mathrm{E}_\alpha$, $[\mathrm{E}_\alpha, \mathrm{E}_{-\alpha}] = \alpha^i \mathrm{H}_i$, $[\mathrm{E}_\alpha, \mathrm{E}_\beta] = N_{\alpha\beta}\mathrm{E}_{\alpha+\beta})$ whose interpretation will be a symmetry group of a quantum hamiltonian (acting in a unitary irreducible representation Λ on the Hilbert space V_Λ) one may define a set of coherent states by choosing a particular state $|0\rangle$ (with $\langle 0|\,|0\rangle = 1$) in V_Λ and acting on it by the elements of G. A subroup H of G that leaves $|0\rangle$ invariant up to a phase $(\Lambda(h)\,|0\rangle = e^{i\phi(h)}\,|0\rangle)$ is called maximum stability subgroup. One may then define the coset space G/H elements of which ($g = \omega h$, $h \in H$, $\omega \in G/H$, $\Lambda(g) = \Lambda(\omega)\Lambda(h)$) will parametrize the coherent states, $|\omega, \Lambda\rangle = \Lambda(\omega)\,|0\rangle$.

This definition depends on a choice of group G, its representation Λ and the vector $|0\rangle$. It is natural to assume also that $|0\rangle$ is an eigenstate of the hamiltonian H, e.g., a ground state. For a unitary representation Λ we may choose $\mathrm{H}_i^\dagger = \mathrm{H}_i$, $\mathrm{E}_\alpha^\dagger = \mathrm{E}_{-\alpha}$ and select $|0\rangle$ to be the highest-weight vector of the representation Λ, i.e. demand that it is annihilated by "raising" generators and is an eigen-state of the Cartan generators: (i) $\mathrm{E}_\alpha\,|0\rangle = 0$ for all positive roots α; (ii) $\mathrm{H}_i\,|0\rangle = h_i\,|0\rangle$. In addition, we may demand that $|0\rangle$ is annihilated also by some "lowering" generators, i.e. (iii) $\mathrm{E}_{-\beta}\,|0\rangle = 0$ for *some* negative roots β; the remaining negative roots will be denoted by γ. Then the coherent states are given by

$$|\omega, \Lambda\rangle = \exp\left[\sum_\gamma (w_\gamma E_{-\gamma} - w_\gamma^* E_\gamma)\right]\,|0\rangle\ , \tag{A.1}$$

where γ are the negative roots for which $\mathrm{E}_\gamma\,|0\rangle \neq 0$. w_γ may be interpreted as coordinates on G/H where H is generated by $(\mathrm{H}_i, \mathrm{E}_\alpha, \mathrm{E}_{-\beta})$.

For example, in the case of $G = \mathrm{SU}(3)$ with the Cartan basis

$$(\mathrm{H}_1, \mathrm{H}_2, \mathrm{E}_\alpha, \mathrm{E}_\beta, \mathrm{E}_{\alpha+\beta}, \mathrm{E}_{-\alpha}, \mathrm{E}_{-\beta}, \mathrm{E}_{-\alpha-\beta})$$

and with $|0\rangle$ being the highest-weight of the fundamental representation (discussed in section 3.2), i.e. $\mathrm{E}_{-\beta}\,|0\rangle = 0$, $\mathrm{E}_{-\alpha}\,|0\rangle \neq 0$, $\mathrm{E}_{-\alpha-\beta}\,|0\rangle \neq 0$, the subgroup H is generated by $(\mathrm{H}_1, \mathrm{H}_2, \mathrm{E}_\beta, \mathrm{E}_{-\beta})$, i.e. is $\mathrm{SU}(2) \times \mathrm{U}(1)$ and $G/H = \mathrm{SU}(3)/(\mathrm{SU}(2) \times \mathrm{U}(1)) = CP^2$.

In section 5 we are interested in the case of the fundamental representation of SO(6). Then we may write $|0\rangle$ as a linear superposition of the highest-weight states of the fundamental irreducible representation invariant under a maximal subgroup H of SO(6). There are four such maximal subgroups for SO(6): SO(5), SO(3) $\times$ SO(3), SO(4) $\times$ SO(2) and SU(3) $\times$ U(1). The first (last) two contain two (three) Cartan generators of SO(6). It is easy to write down the generators of the first three subgroups in terms of the 15 generators $M_{ij} = (\delta_i^a \delta_j^b - \delta_j^a \delta_i^b)$ in the fundamental representation of SO(6):

$$\begin{aligned}
\mathrm{SO}(5) &= \{M_{ij} : i,j = 1, \ldots, 5\}\,, \\
\mathrm{SO}(3) \times \mathrm{SO}(3) &= \{M_{ij} : i,j = 1,2,3\} \cup \{M_{ij} : i,j = 4,5,6\}\,, \\
\mathrm{SO}(4) \times \mathrm{SO}(2) &= \{M_{ij} : i,j = 1, \ldots, 4\} \cup \{M_{56}\}\,.
\end{aligned} \tag{A.2}$$

To obtain the SU(3) $\times$ U(1) subgroup it is convenient to use the Cartan basis of the SO(6) algebra. The 3 Cartan generators may be chosen as (linear combinations of) M_{12}, M_{34}, M_{56}. The non-Cartan elements are (up to normalisation constants)

$$\begin{aligned}
\mathrm{E}_\alpha &= M_{13} - iM_{14} - iM_{23} - M_{24}\,, & \mathrm{E}_\beta &= M_{15} - iM_{16} + iM_{25} + M_{26}\,, \\
\mathrm{E}_\gamma &= M_{35} - iM_{36} + iM_{45} + M_{46}\,, & \mathrm{E}_{\alpha+\beta} &= M_{35} - iM_{36} - iM_{45} - M_{46}\,, \\
\mathrm{E}_{\alpha+\gamma} &= M_{15} - iM_{16} - iM_{25} - M_{26}\,, & \mathrm{E}_{\gamma-\beta} &= M_{13} + iM_{14} - iM_{23} + M_{24}\,,
\end{aligned} \tag{A.3}$$

with the negative roots $\mathrm{E}_{-\alpha} = (\mathrm{E}_{-\alpha})^*$, *etc.* Then SU(3) $\times$ U(1) is generated by the three Cartan generators together with

$$\mathrm{E}_{\pm\beta}\,, \mathrm{E}_{\pm\gamma}\,, \qquad \mathrm{E}_{\pm(\gamma-\beta)}\,. \tag{A.4}$$

Explicitly, the Cartan generators associated to β and γ are proportional to

$$\mathrm{H}_\beta = M_{12} - M_{34}\,, \qquad \mathrm{H}_\gamma = M_{12} + M_{34} - 2M_{56}\,, \tag{A.5}$$

with the U(1) generator $\mathrm{H}_{\mathrm{U}(1)}$ proportional to $M_{12} + M_{34} + M_{56}$.

Let us now identify suitable vacua. For $H = \mathrm{SO}(5)$ we have

$$|0\rangle_{\mathrm{SO}(5)} = (0,0,0,0,0,1)\,, \tag{A.6}$$

since then $M_{ij}\,|0\rangle_{\mathrm{SO}(5)} = 0$ for $i,j = 1,\ldots,5$. For $H = \mathrm{SO}(4)\times\mathrm{SO}(2)$ we have instead

$$|0\rangle_{\mathrm{SO}(4)\times\mathrm{SO}(2)} = (0,0,0,0,1,i)\,. \tag{A.7}$$

In the case of $H = \mathrm{SO}(3)\times\mathrm{SO}(3)$ subgroup the Cartan generators may be chosen as M_{12} and M_{56} which have four eigenvectors

$$(i,1,0,0,0,0)\,, \qquad (-i,1,0,0,0,0)\,, \qquad (0,0,0,0,i,1)\,, \qquad (0,0,0,0,-i,1)\,,$$

whose eigenvalues are $(\mp i, 0)$ and $(0, \pm i)$, and two eigenvectors

$$(0,0,1,0,0,0)\,, \qquad (0,0,0,1,0,0)$$

with zero eigenvalues. Clearly, the first four eigenvectors are not annihilated by the non-Cartan part of $\mathrm{SO}(3)\times\mathrm{SO}(3)$ (that is by M_{13}, M_{23}, M_{45} and M_{46}). It is also easy to check that no linear combination of the latter two eigenvectors is annihilated by the non-Cartan part of $\mathrm{SO}(3)\times\mathrm{SO}(3)$. We conclude that, in the fundamental representation of SO(6), there is no $\mathrm{SO}(3)\times\mathrm{SO}(3)$ invariant vacuum state.

Similarly for $H = \mathrm{SU}(3)\times\mathrm{U}(1)$ there is no invariant vacuum state, in the anti-symmetric (six-dimensional) representation of SU(4). The eigenvectors of the Cartan generators are:

$$(\mp i,1,0,0,0,0)\,, \qquad (0,0,\pm i,1,0,0)\,, \qquad (0,0,0,0,\pm i,1)\,. \tag{A.8}$$

Their $(\mathrm{H}_\beta, \mathrm{H}_\gamma, \mathrm{H}_{\mathrm{U}(1)})$ weights are $\pm(i,i,i)$, $\pm(i,-i,-i)$ and $\pm(0,2i,-i)$, respectively. It is not difficult to see that none of the above six states is annihilated by all of the non-Cartan generators (A.4). This conclusion is clearly representation dependent; for example, in the fundamental (four-dimensional) representation of SU(4) we can easily find a suitable ground state: this is just $(0,0,0,1)$.

B. Coherent state expectation value of SL(2) spin chain hamiltonian

Our aim here is to compute the expectation value $\langle \ell_{l,l+1}|\,H_{l,l+1}\,|\ell_{l,l+1}\rangle$ in (4.19). Let us set for notational simplicity $l = 1,\ l+1 = 2$ and define

$$\mathcal{M} \equiv \frac{\langle \ell_{12}|\,H_{12}\,|\ell_{12}\rangle}{(1-|\zeta_1|^2)(1-|\zeta_2|^2)}\,. \tag{B.1}$$

One finds using (4.10)

$$\begin{aligned}
\mathcal{M} &= \sum_{n_1,n_2,m_1,m_2=0}^{\infty} \frac{\zeta_1^{n_1}\zeta_2^{n_2}\bar{\zeta}_1^{m_1}\bar{\zeta}_2^{m_2}}{m_1!m_2!} \langle 0|\, a_1^{m_1} a_2^{m_2} H_{12} a_1^{\dagger n_1} a_2^{\dagger n_2}\, |0\rangle \\
&= \sum_{m_1,m_2=0}^{\infty} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\zeta_1^{k}\zeta_2^{n-k}\bar{\zeta}_1^{m_1}\bar{\zeta}_2^{m_2}}{m_1!m_2!} \langle 0|\, a_1^{m_1} a_2^{m_2} H_{12} a_1^{\dagger k} a_2^{\dagger n-k}\, |0\rangle \\
&= \sum_{m_1,m_2=0}^{\infty} \sum_{n=0}^{\infty}\sum_{k,l=0}^{n} \frac{\zeta_1^{k}\zeta_2^{n-k}\bar{\zeta}_1^{m_1}\bar{\zeta}_2^{m_2}}{m_1!m_2!} c_l(k,n)\, \langle 0|\, a_1^{m_1} a_2^{m_2} H_{12} a_1^{\dagger l} a_2^{\dagger n-l}\, |0\rangle \\
&= \sum_{n=0}^{\infty}\sum_{k,l=0}^{n} \zeta_1^{k}\zeta_2^{n-k}\bar{\zeta}_1^{l}\bar{\zeta}_2^{n-l} c_l(k,n)\,,
\end{aligned} \tag{B.2}$$

i.e.

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \bar{\mathcal{M}}_2\,, \tag{B.3}$$

where

$$\mathcal{M}_1 = \bar{\mathcal{M}}_1 = \sum_{n=0}^{\infty}\sum_{k=0}^{n} |\zeta_1|^{2k}|\zeta_2|^{2n-2k}[h(k)+h(n-k)]\,, \tag{B.4}$$

$$\mathcal{M}_2 = -\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k-1} \zeta_1^{k}\zeta_2^{n-k}\bar{\zeta}_1^{l}\bar{\zeta}_2^{n-l}\frac{1}{k-l}\,. \tag{B.5}$$

Computing these terms explicitly we get

$$\begin{aligned}
\mathcal{M}_1 &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} |\zeta_1|^{2k}|\zeta_2|^{2n-2k}[h(k)+h(n-k)] \\
&= \sum_{m_1,m_2=0}^{\infty} |\zeta_1|^{2m_1}|\zeta_2|^{2m_2}[h(m_1)+h(m_2)] \\
&= \frac{1}{1-|\zeta_1|^2}\sum_{m_2=0}^{\infty} |\zeta_2|^{2m_2}h(m_2) + \frac{1}{1-|\zeta_2|^2}\sum_{m_1=0}^{\infty}|\zeta_1|^{2m_1}h(m_1) \\
&= -\frac{\ln(1-|\zeta_2|^2)+\ln(1-|\zeta_1|^2)}{(1-|\zeta_1|^2)(1-|\zeta_2|^2)}\,,
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
\mathcal{M}_2 &= -\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k-1} \zeta_1^{k}\zeta_2^{n-k}\bar{\zeta}_1^{l}\bar{\zeta}_2^{n-l}\frac{1}{k-l} \\
&= -\sum_{m_1,m_2=0}^{\infty}\sum_{l=0}^{m_1-1} \zeta_1^{m_1}\zeta_2^{m_2}\bar{\zeta}_1^{l}\bar{\zeta}_2^{m_1+m_2-l}\frac{1}{m_1-l} \\
&= -\frac{1}{1-|\zeta_2|^2}\sum_{m_1=0}^{\infty}\sum_{l=0}^{m_1-1} \zeta_1^{m_1}\bar{\zeta}_1^{l}\bar{\zeta}_2^{m_1-l}\frac{1}{m_1-l} \\
&= -\frac{1}{1-|\zeta_2|^2}\sum_{n_1,n_2=0}^{\infty} \zeta_1^{n_1+n_2+1}\bar{\zeta}_1^{n_1}\bar{\zeta}_2^{n_2+1}\frac{1}{n_2+1}
\end{aligned}$$

$$= -\frac{1}{(1-|\zeta_2|^2)(1-|\zeta_1|^2)} \sum_{n_2=0}^{\infty} (\zeta_1\bar{\zeta}_2)^{n_2+1} \frac{1}{n_2+1}$$

$$= \frac{\ln(1-\zeta_1\bar{\zeta}_2)}{(1-|\zeta_2|^2)(1-|\zeta_1|^2)}\,, \tag{B.7}$$

and thus also

$$\bar{\mathcal{M}}_2 = \frac{\ln(1-\zeta_2\bar{\zeta}_1)}{(1-|\zeta_2|^2)(1-|\zeta_1|^2)}\,. \tag{B.8}$$

As a result, we finish with

$$\langle \ell_{12}|\, H_{12}\, |\ell_{12}\rangle = \ln \frac{(1-\zeta_1\bar{\zeta}_2)(1-\zeta_2\bar{\zeta}_1)}{(1-|\zeta_2|^2)(1-|\zeta_1|^2)}\,, \tag{B.9}$$

or, equivalently, using (4.18),

$$\langle \ell_{12}|\, H_{12}\, |\ell_{12}\rangle = \ln \frac{1-\eta_{ij}\ell_1^i\ell_2^j}{2}\,. \tag{B.10}$$

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